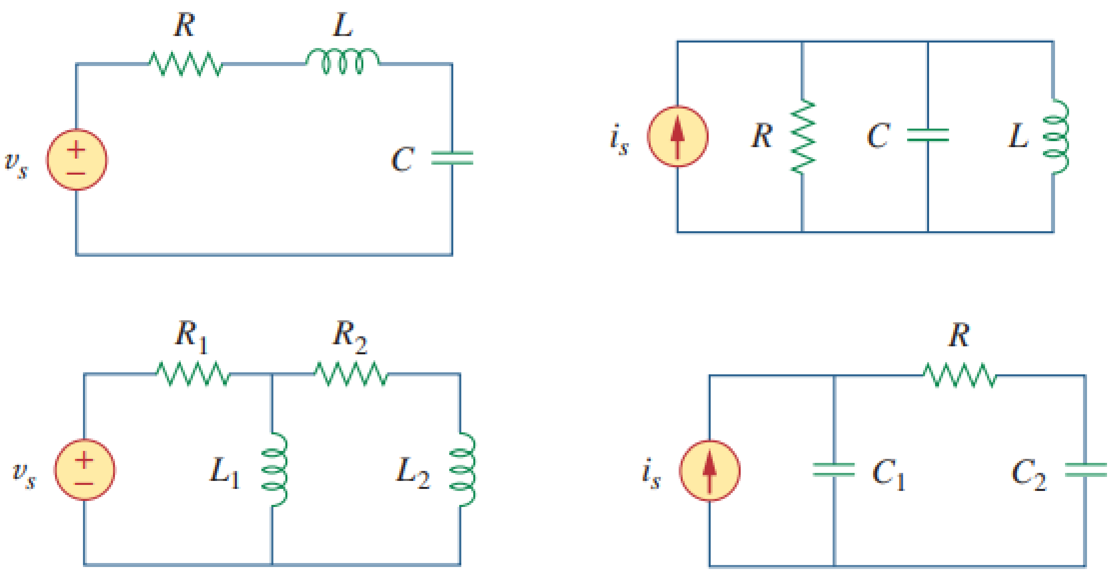
3.3. Second order transient circuits

In the previous chapter we considered circuits with a single storage element (a capacitor or an inductor). Such circuits are first-order because the differential equations describing them are first-order. In this chapter we will consider circuits containing two storage elements. These are known as *second-order* circuits because their responses are described by differential equations that contain second derivatives. Typical examples of second-order circuits are *RLC* circuits, in which the three kinds of passive elements are present.



Generally, second-order circuit may have two storage elements of different type or the same type (provided elements of the same type cannot be represented by an equivalent single element).

A second-order circuit is characterized by a second-order differential equation. It consists of resistors and the equivalent of two energy storage elements. Given a second-order circuit, we can determine its step response x(t) (which may be voltage or current) by taking the following four steps:

1. **We first determine the initial conditions $\mathbf{x(0)}$ and $\mathbf{dx(0)/dt}$ and the final value $\mathbf{x(\infty)}$.**

There are two key points to keep in mind in determining the initial conditions.

a. First-as always in circuit analysis—we must carefully handle the polarity of voltage *v(t)* across the capacitor and the direction of the current *i(t)* through the inductor.
b. The capacitor voltage and the inductor current are always continuous so that, $v(0^-) = v(0^+)$ and so that, $i(0^-) = i(0^+)$ respectively.

Where $t = 0^-$ denotes the time just before a switching event and $t = 0^+$ is the time just after the switching event, assuming that the switching event takes place at $t = 0$. Thus, in finding initial conditions, we first focus on those variables that cannot change abruptly, capacitor voltage and inductor current.

Exercise 3.1

The switch in Fig. 1 has been closed for a long time. It is open at t = 0. Find: (a) $i(0^+), v(0^+)$, (b) $di(0^+)/dt, dv(0^+)/dt$, (c) $i(\infty), v(\infty)$.

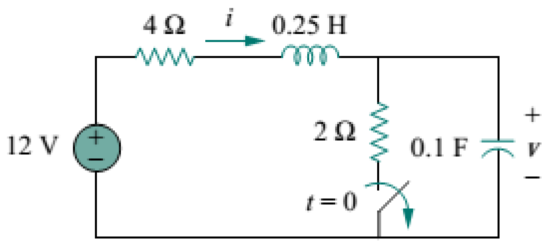


Figure 1

2. **We find the natural response $x_n(t)$ by turning off independent Sources and applying KCL and KVL.**

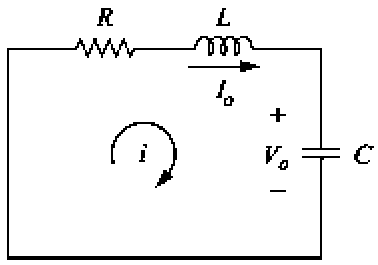
a) Source-free series RLC circuit

The circuit is being excited by the energy initially stored in the capacitor and inductor. The energy is represented by the initial capacitor voltage V_o and initial inductor current I_o. Thus, at t = 0,

$$v(0) = \frac{1}{C}\int_{-\infty}^{0} i\,dt = V_0 \text{ and } i(0) = I_0$$

➢ Applying KVL around the loop

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{0} i\,dt = 0$$



Differentiating both side

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$$

or Applying KVL around the loop

$$Ri + L\frac{di}{dt} + v_c = 0$$

But, $i_c = \frac{1}{C}\frac{dv}{dt}$ $$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{v}{LC} = 0$$

Both equation $\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$ and $\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{v}{LC} = 0$

Represent series RLC circuit

To solve such a second-order differential equation requires that we have two initial conditions, such as the initial value of *i* and its first derivative.

$$i(0) = I_0 \text{ and } \frac{di(0)}{dt} = -\frac{1}{L}(Ri(0) + v_c(0))$$

$$v(0) = V_0 \text{ and } \frac{dv(0)}{dt} = \frac{I_L(0)}{C}$$

- With the two initial conditions it possible for solving SODE

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$$

From first-order circuits suggests that the solution is of exponential form. So we let

$$i(t) = Ae^{st}$$

Where A and s are constants to be determined.

By substituting $i(t) = Ae^{st}$ into $\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$

$$AS^2e^{st} + \frac{AR}{L}se^{st} + \frac{A}{LC}e^{st} = 0$$

$$Ae^{st}(S^2 + \frac{R}{L}s + \frac{1}{LC}) = 0$$

Since $i(t) = Ae^{st}$ is the assumed solution we are trying to find, only the expression in parentheses can be zero:

$$S^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

This quadratic equation is known as the characteristic equation of the given SODE and the solution of characteristic equation given by

$$S_{1,2} = \frac{(-b \pm \sqrt{b^2 - 4ac})}{2a}$$

$$S_1 = \frac{-R}{2L} + \sqrt{(\frac{R}{2L})^2 - \frac{1}{LC}} \quad \text{and } S_2 = \frac{-R}{2L} - \sqrt{(\frac{R}{2L})^2 - \frac{1}{LC}}$$

A more compact way of expressing the roots is

$$S_1 = -\boldsymbol{\alpha} + \sqrt{\boldsymbol{\alpha}^2 - w_0^{\,2}} \text{ and } S_2 = -\boldsymbol{\alpha} - \sqrt{\boldsymbol{\alpha}^2 - w_0^{\,2}}$$

Where,

$$\boldsymbol{\alpha} = \frac{R}{2L} \text{ and } w_0 = \frac{1}{\sqrt{LC}}$$

The roots S_1 and S_2 are called *natural frequencies*, measured in nepers per second (Np/s), because they are associated with the natural response of the circuit; ω_0 is known as the *resonant frequency* or strictly as the *undamped natural frequency*, expressed in radians per second (rad/s); and α is the *neper frequency* or the *damping factor*, expressed in nepers per second.

- In terms of ω_0 and α

$$S^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \qquad S^2 + 2\boldsymbol{\alpha}S + w_0^{\,2} = 0$$

The two values of *s* indicate that there are two possible solutions for *i*, each of which is of the form of the assumed solution

$$i_1(t) = A_1 e^{S_1 t} \qquad i_2(t) = A_2 e^{S_2 t}$$

Since $\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$ is a linear equation, any linear combination of the two distinct solutions and is also a solution of DE. A complete or total solution of DE would therefore require a linear combination of i_1 and i_2. Thus, the natural response of the series *RLC* circuit is

$$i(t) = A_1 e^{S_1 t} + A_2 e^{S_2 t}$$

Where A_1 and A_2 the constants and are determined from the initial values $i(0^+)$ $di(0^+)/dt$

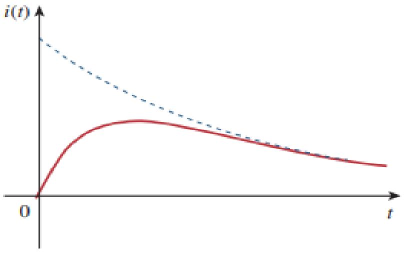
- We can infer that there are three types of solutions:
  1. If $\alpha > \omega_0$ we have the *over damped* case.
  2. If $\alpha = \omega_0$ we have the *critically damped* case.
  3. If $\alpha < \omega_0$ we have the *under damped* case.

I. **If α > ω_0 we have the *over damped* case.**

- **α > ω_0** implies C > 4L/ R^2. When this happens, both roots S_1 and S_2 are negative and real. The response is

$$i(t) = A_1 e^{S_1 t} + A_2 e^{S_2 t}$$

This indicates source free response decays and approaches to zero as t increases. Typical over damped response is illustrated as follows.



II. **If α= ω_0 we have the *critically damped* case.**

- **α > ω_0** implies C = 4L/ R^2. Then $S_1 = S_2 = -\alpha = -R/2L$

For this case
$$i(t) = A_1 e^{-\alpha t} + A_2 e^{-\alpha t} = A_3 e^{-\alpha t}$$

Where $A_3 = A_1 + A_2$. This cannot be the solution, because the two initial conditions cannot be satisfied with the single constant. So, our assumption of an exponential solution is incorrect for the special case of critical damping.

$$\frac{d^2 i}{dt^2} + 2\alpha \frac{di}{dt} + \alpha^2 i = 0$$

$$\frac{d}{dt}\left(\frac{di}{dt} + \alpha i\right) + \alpha\left(\frac{di}{dt} + \alpha i\right) = 0$$ If we let $f = \frac{di}{dt} + \alpha i$

$$\frac{df}{dt} + \alpha f = 0$$ which is a first-order differential equation with solution $f = A_1 e^{-\alpha t}$ where is A_1 constant.

$$\frac{di}{dt} + \alpha i = A_1 e^{-\alpha t} \qquad e^{\alpha t}\frac{di}{dt} + \alpha e^{\alpha t} i = A_1$$

$\frac{d}{dt}(ie^{\alpha t}) = A_1$ Integrating both sides yields $ie^{\alpha t} = A_1 t + A_2$

$$i(t) = (A_1 t + A_2)\, e^{-\alpha t}$$

Hence, the natural response of the critically damped circuit is a sum of two terms: a negative exponential and a negative exponential multiplied by a linear term.

III. If $\alpha < \omega_0$ we have the *under damped* case.

- $\boldsymbol{\alpha} < \boldsymbol{\omega_0}$ implies $C < 4L/R^2$. The roots may be written as

$$S_1 = -\boldsymbol{\alpha} + \sqrt{-(w_0{}^2 - \boldsymbol{\alpha}^2)} \text{ and } S_2 = -\boldsymbol{\alpha} - \sqrt{-(w_0{}^2 - \boldsymbol{\alpha}^2)}$$

or

$$S_1 = -\boldsymbol{\alpha} + jw_d \text{ and } S_2 = -\boldsymbol{\alpha} - jw_d$$

- $w_d = \sqrt{(w_0{}^2 - \boldsymbol{\alpha}^2)}$ which is called the damping frequency.

$$\text{The natural response } i(t) = A_1 e^{(-\boldsymbol{\alpha}+jw_d)t} + A_2 e^{(-\boldsymbol{\alpha}-jw_d)t}$$

$$i(t) = e^{-\boldsymbol{\alpha}t}(A_1 e^{jw_d t} + A_2 e^{-jw_d t})$$

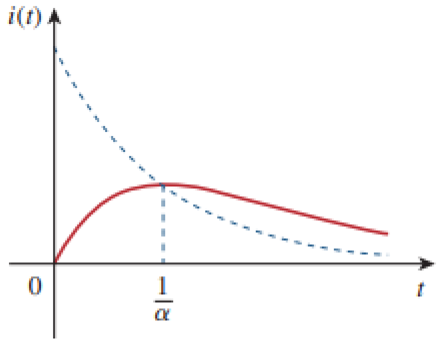
Using Euler's identities, $e^{j\theta} = \cos\theta + j\sin\theta$ and $e^{-j\theta} = \cos\theta - j\sin\theta$

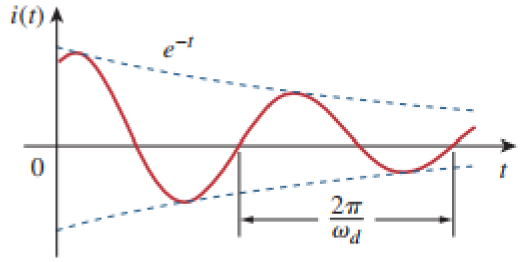
$$i(t) = e^{-\boldsymbol{\alpha}t}(A_1(\cos w_d t + j\sin w_d t) + A_2(\cos w_d t + j\sin w_d t))$$

$$i(t) = e^{-\boldsymbol{\alpha}t}((A_1 + A_2)\cos w_d t + j(A_1 - A_2)\sin w_d t)$$

$$i(t) = e^{-\boldsymbol{\alpha}t}(B_1 \cos w_d t + jB_2 \sin w_d t)$$

- With the presence of sine and cosine functions, it is clear that the natural response for this case is exponentially damped and oscillatory in nature.





The waveforms of the responses for critically damped case and under damped case.

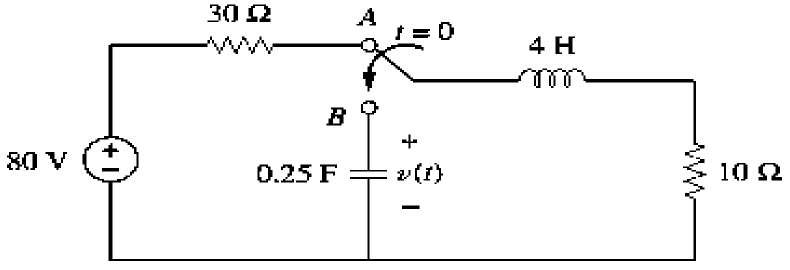
- The damping effect is due to the presence of resistance *R*. The damping factor determines the rate at which the response is damped.
- Oscillatory response is possible due to the presence of the two types of storage elements. Having both *L* and *C* allows the flow of energy back and forth between the two. The damped oscillation exhibited by the under damped response is known as *ringing*. It stems from the ability of the storage elements *L* and *C* to transfer energy back and forth between them.
- The critically damped case is the borderline between the under damped and over damped cases and it decays the fastest.

- Both ω_0 and ω_d are natural frequencies because they help determine the natural response; ω_0 is often called the *undamped natural frequency* and ω_d is called the *damped natural frequency*.

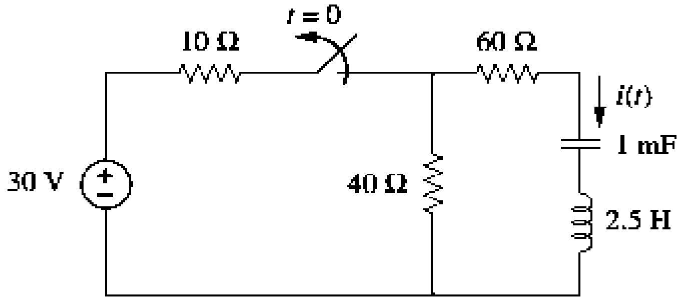
Example 1 If R= 50 Ω, L=1.5 H, what value of *C* will make an *RLC* series circuit:

a. Over damped,
b. critically damped
c. Under damped?

Example 2 The switch in Fig. bellow moves from position *A* to position *B.* find v(t) for *t>0*.



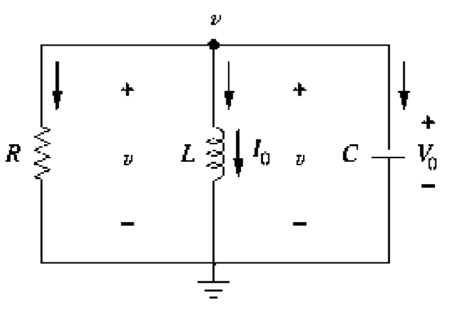
Exercise: For the circuit shown below, Find i(t) for all *t* >0.



b). the Source-Free Parallel *RLC* Circuit

Parallel *RLC* circuits find many practical applications, notably in communications networks and filter designs. Consider the parallel RLC the circuit is excited by the energy initially stored in capacitor and inductor. The energy represented by initial capacitor Voltage V_o and initial inductor current I_o.

$$i(0) = \frac{1}{L}\int_{-\infty}^{0} v(t)\,dt = I_0 \text{ and } v(0) = v_0$$



Because the three elements are in parallel, they have the same voltage *v* across them. According to passive sign convention, the current is entering each element; that is, the current through each element is leaving the top node. Thus, applying KCL at the top node gives

$$\frac{v}{R} + \frac{1}{L}\int_{-\infty}^{0} v(t)\,dt + C\frac{dv}{dt} = 0$$

Taking the derivative with respect to *t* and dividing by *C* results in

$$\frac{d^2v}{dt^2} + \frac{1}{Rc}\frac{dv}{dt} + \frac{v}{LC} = 0 \qquad v(0) = v_0$$

To find the initial conditions

$$\frac{v}{R} + \mathrm{i_L} + C\frac{dv}{dt} = 0 \qquad \frac{v(0)}{R} + \mathrm{i_L}(0) + C\frac{dv(0)}{dt} = 0$$

$$\frac{d^2v}{dt^2} + \frac{1}{Rc}\frac{dv}{dt} + \frac{v}{LC} = 0 \quad \begin{cases} \frac{dv(0)}{dt} = \frac{-1}{C}\left(\frac{v(0)}{R} + \mathrm{i_L}(0)\right) \\ v(0) = v_0 \end{cases}$$

Or from KCL $\mathrm{i_R} + \mathrm{i_L} + \mathrm{i_C} = 0$

$$\frac{v}{R} + \mathrm{i_L} + C\frac{dv}{dt} = 0 \text{ but } v_R = v_L = v_C = L\frac{di}{dt}$$

$$\frac{d^2i}{dt^2} + \frac{1}{Rc}\frac{di}{dt} + \frac{i}{LC} = 0 \quad \begin{cases} \mathrm{I}(0) = I_0 \\ \frac{\mathrm{d}i(0)}{\mathrm{dt}} = \frac{V_c(0)}{\mathrm{C}} \end{cases}$$

We obtain the characteristic equation by replacing the first derivative by *s* and the second derivative by s^2.

$$S^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

The roots of the characteristic equation are

$$S_1 = \frac{-1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}} \quad \text{and } S_2 = \frac{-1}{2RC} - \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

In a compact way

$$\mathrm{S_1} = -\boldsymbol{\alpha} + \sqrt{\boldsymbol{\alpha}^2 - \mathrm{w_0}^2} \text{ and } \mathrm{S_2} = -\boldsymbol{\alpha} - \sqrt{\boldsymbol{\alpha}^2 - \mathrm{w_0}^2}$$

$$\boldsymbol{\alpha} = \frac{1}{2RC} \text{ and } \mathrm{w_0} = \frac{1}{\sqrt{LC}}$$

From $\mathrm{S_1} = -\boldsymbol{\alpha} + \sqrt{\boldsymbol{\alpha}^2 - \mathrm{w_0}^2}$ and $\mathrm{S_2} = -\boldsymbol{\alpha} - \sqrt{\boldsymbol{\alpha}^2 - \mathrm{w_0}^2}$ we can infer that there are three types of solutions.

I. If $\alpha > \omega_0$ we have the *over damped* case.

- $\boldsymbol{\alpha > \omega_0}$ implies $L > 4CR^2$. When this happens, both roots S_1 and S_2 are negative and real. The response is $v(t) = A_1 e^{S_1 t} + A_2 e^{S_2 t}$

II. **If $\alpha = \omega_0$ we have the *critically damped* case.**

$\alpha = \omega_0$ implies $L = 4CR^2$. Then $S_1 = S_2 = -\alpha = -1/2RC$

$$v(t) = (A_1 t + A_2)\, e^{-\alpha t}$$

III. **If $\alpha < \omega_0$ we have the *under damped* case.**

- $\alpha < \omega_0$ implies $L = 4CR^2$. The roots may be written as

$$S_1 = -\alpha + \sqrt{-(w_0^2 - \alpha^2)} \text{ and } S_2 = -\alpha - \sqrt{-(w_0^2 - \alpha^2)}$$

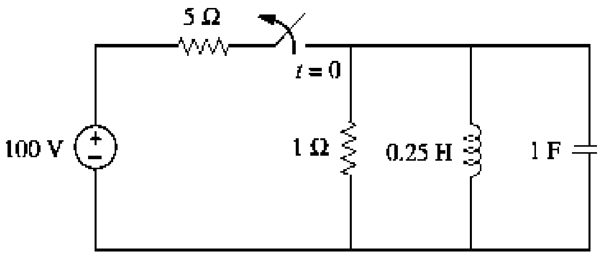
or

$$S_1 = -\alpha + jw_d \text{ and } S_2 = -\alpha - jw_d$$

The response is

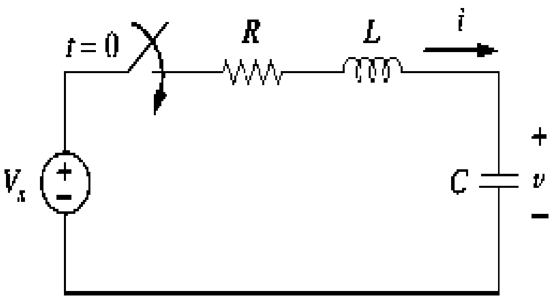
$$v(t) = e^{-\alpha t}(B_1 \cos w_d t + jB_2 \sin w_d t)$$

Example 3: Find the voltage across the capacitor as a function of time for the circuit shown below. Assume steady-state conditions exist at *t*= 0-.



c. **Step Response of a Series RLC Circuit**

- The step response is obtained by the sudden application of a dc source. Consider the series *RLC* circuit shown in figure bellow. Applying KVL around the loop for t>0 ,



$$L\frac{di}{dt} + Ri + \frac{1}{C}\int idt = v_s$$

$$\frac{d^2i}{d^2t} + \frac{R}{L}\frac{dv}{dt} + \frac{i}{LC} = 0$$

or

$$L\frac{di}{dt} + Ri + v = v_s \qquad i = C\frac{dv}{dt} \quad \Rightarrow \quad \frac{dv^2}{dt} + \frac{R}{L}\frac{dv}{dt}i + \frac{v}{CL} = \frac{v_s}{CL}$$

$$v(0) = V_0 \text{ and } \frac{dv(0)}{dt} = \frac{I_L(0)}{C}$$

- The solution to SODE has two components: the transient response and the steady-state response

$$v(t) = v_t(t) + v_{ss}(t)$$

The transient response is the component of the total response that dies out with time.

(Over damped) ➡ $\alpha > w_o$ ➡ $v(t) = A_1 e^{S_1 t} + A_2 e^{S_2 t}$

(Critically damped) ➡ $\alpha = w_o$ ➡ $v(t) = (A_1 t + A_2)\, e^{-\alpha t}$

(Under damped) ➡ $\alpha < w_o$ ➡ $V(t) = e^{-\alpha t}(B_1 cos w_d t + B_2 sin w_d t)$

- The steady-state response is the final value of v(t). For series RLC circuit, the final value of the capacitor voltage is the same as the source voltage. Hence,

$$v_{ss}(t) = v(\infty) = V_s$$

Thus, the complete solutions for the over damped, under damped, and critically damped cases are:

$$v(t) = V_s + A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad \text{(Overdamped)}$$

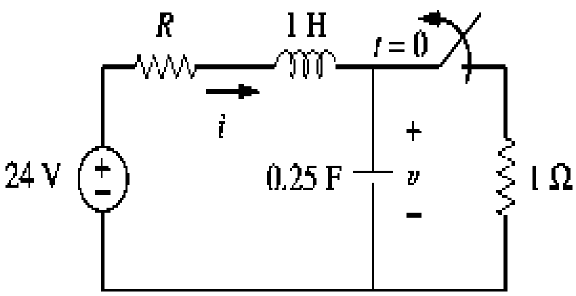
$$v(t) = V_s + (A_1 + A_2 t)e^{-\alpha t} \quad \text{(Critically damped)}$$

$$v(t) = V_s + (A_1 \cos \omega_d t + A_2 \sin \omega_d t)e^{-\alpha t} \quad \text{(Underdamped)}$$

The values of the constants A_1 and A_2 are obtained from the initial conditions: v(0) and $dv(0)/dt$. Keep in mind that v and i are, respectively, the voltage across the capacitor and the current through the inductor.

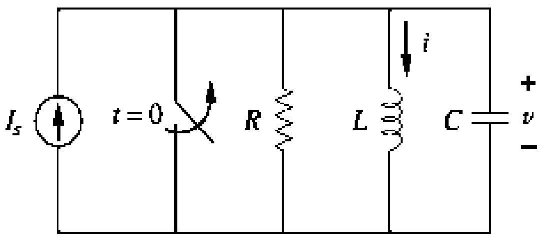
But once the capacitor voltage $v_c = v$ is known, we can determine $i = Cdv/dt$, which is the same current through the capacitor, inductor, and resistor. Hence, the voltage across the resistor is $v_R = iR$, while the inductor voltage is $V_L = Ldi/dt$.

Example 4: Find the voltage across the capacitor as a function of time for the circuit shown below. Assume steady-state conditions exist at t= 0-. If R= 5 ohms



d. **Step Response of a Parallel RLC Circuit**

- Consider the parallel RLC circuit shown in figure bellow. We want to find i(t) due to a sudden application of a dc current. Applying KCL at the top node for t >0.



$$\frac{v}{R} + C\frac{dv}{dt} + \frac{1}{L}\int v\,dt = I_s$$

$$\frac{1}{R}\frac{dv}{dt} + C\frac{d^2v}{d^2t} + \frac{V}{L} = 0$$

$$\frac{d^2v}{d^2t} + \frac{1}{CR}\frac{dv}{dt} + \frac{V}{LC} = 0$$

Or

$$\frac{v}{R} + C\frac{dv}{dt} + i = I_s \qquad v = L\frac{di}{dt} \quad \Rightarrow \quad \frac{di^2}{dt} + \frac{1}{RC}\frac{di}{dt} + \frac{i}{LC} = \frac{I_s}{LC}$$

$$\mathrm{I}(0) = I_0 \qquad \frac{\mathrm{d}i(0)}{\mathrm{dt}} = \frac{V_c(0)}{\mathrm{C}}$$

The complete solution of parallel RLC has two components: the transient response and the steady-state response

$$i(t) = i_t(t) + i_{ss}(t)$$

The transient response is the same as what we had in source free parallel RLC

$$i_t(t) = A_1e^{S_1t} + A_2e^{S_2t} \qquad \text{(Over damped)}$$

$$i_t(t) = (A_1 + A_2t)e^{-\alpha t} \qquad \text{(Critically damped)}$$

$$i_t(t) = e^{-\alpha t}(A_1cosw_dt + A_2sinw_dt) \qquad \text{(Under damped)}$$

The steady-state response is the final value of *i*. In the circuit in step response of parallel RLC, the final value of the current through the inductor is the same as the source current *Is*. Thus,

$$i(t) = I_s + A_1e^{s_1t} + A_2e^{s_2t} \qquad \text{(Overdamped)}$$

$$i(t) = I_s + (A_1 + A_2t)e^{-\alpha t} \qquad \text{(Critically damped)}$$

$$i(t) = I_s + (A_1\cos\omega_dt + A_2\sin\omega_dt)e^{-\alpha t} \qquad \text{(Underdamped)}$$

Example 5: For the circuit shown below, find i(t) and $i_R(t)$ for t >0.